

The University of Chicago
FOUNDED BY JOHN D. ROCKEFELLER

Group-Characters of Various Linear Groups

A DISSERTATION

SUBMITTED TO THE FACULTY OF THE OGDEN GRADUATE SCHOOL OF SCIENCE
IN CANDIDACY FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

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HERBERT E. JORDAN

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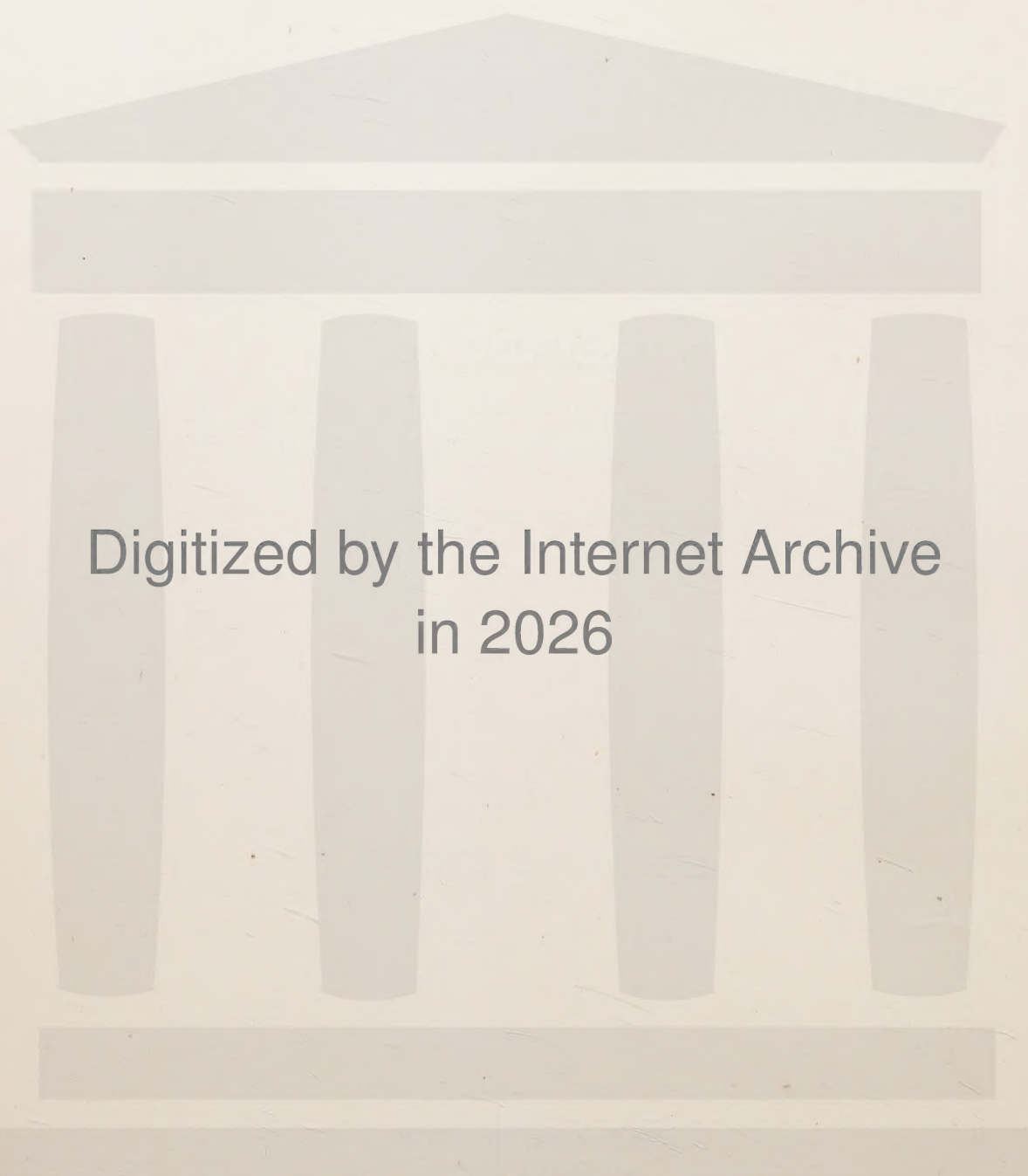
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*Group-Characters of Various Types of Linear Groups.**

BY HERBERT E. JORDAN.

INTRODUCTION.

In an article entitled *Über Gruppencharaktere*† Frobenius has determined the group-characters of the group of all binary linear fractional substitutions of determinant unity (when in their normal forms), the coefficients being taken modulo p , an odd prime. In the present paper the same method is applied to more general types of groups.

In Part I we consider the group $H \equiv SLH(2, p^n)$, $p > 2$, of all binary linear homogeneous substitutions in the $GF[p^n]$, of determinant unity. By the aid of two theorems due to Frobenius on the relation between the characters of a group and those of one of its quotient-groups, we deduce as a corollary the characters of the group $F \equiv LF(2, p^n)$, $p > 2$, of all binary linear fractional substitutions in the $GF[p^n]$ of determinant unity (when in their normal forms). We have also obtained these characters directly by the method applied to the group H ; the chief points of difference in the treatment are stated in foot-notes. The results are a direct generalization of those obtained by Frobenius. In Part II we consider the group $H_1 \equiv SLH(2, p^n)$, $p = 2$. This is identical with the group $LF(2, p^n)$, $p = 2$. Part III deals with the group F_1 of all binary linear fractional substitutions in the $GF[p^n]$, $p > 2$, of determinant not zero. The group H is treated with considerable detail; the others briefly.

Frobenius‡ has determined by another method the group-characters of the

* The abstract of the above paper appeared in the *Bulletin of the American Mathematical Society*, April, 1904. Just recently Schur has computed by different methods the characters of these same types of groups: *Untersuchungen über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen*, Crelle, Vol. 132, 1906-7, (Heft 2).

† *Berliner Sitzungsberichte*, 1896, pp. 985-1021.

‡ *Ueber die Composition der Charaktere einer Gruppe*, *Berliner Sitzungsberichte*, 1899, pp. 330-339.

groups $SLH(2, 3)$, $(2, 5)$, and of the alternating group on six letters of order 360, which is isomorphic with the group $LF(2, 3^2)$ of determinant unity. Burnside* has obtained the group-characters of the binary linear homogeneous group in the $GF[2^3]$ of order 504. The results in this paper agree with those for the above special groups.

I.

The Binary Linear Homogeneous Group H in the $GF[p^n]$, $p > 2$, of Determinant Unity.

§ 1.

The order of the group H is $h = p^n(p^n - 1)$.† For the substitution

$$R: \quad \begin{aligned} x' &= \alpha x + \beta y, \\ y' &= \gamma x + \delta y, \end{aligned} \quad \alpha\delta - \beta\gamma = 1,$$

we use the notation‡ $R = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

We first reduce the substitutions of H to their canonical forms.§ For this purpose we consider the characteristic equation

$$K^2 - K(\alpha + \delta) + 1 = 0 \tag{1}$$

of the substitution $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. If the roots of this equation are distinct we get the canonical form A or B :

$$A: \quad \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix}, \quad \rho \text{ a mark } \neq 0 \text{ of the } GF[p^n],$$

$$B: \quad \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix}, \quad \sigma \text{ a root of } \sigma^{s+1} = 1,$$

according as the equation (1) is reducible or irreducible in the $GF[p^n]$. If the

* Proc. Lond. Math. Soc., Series 2, Vol. I—Part 2, p. 116.

† We shall throughout denote p^n by s , except in the notation $GF[p^n]$.

‡ For the substitution R taken fractionally we use the notation $R = \begin{pmatrix} a & \beta \\ \gamma & \delta \end{pmatrix}$. To the two substitutions $\begin{pmatrix} a & \beta \\ \gamma & \delta \end{pmatrix}$ and $\begin{pmatrix} -a & -\beta \\ -\gamma & -\delta \end{pmatrix}$ of H corresponds the one substitution $\begin{pmatrix} a & \beta \\ \gamma & \delta \end{pmatrix}$ of F . We have therefore a two-to-one correspondence between H and F .

§ Dickson Linear Groups, §§ 214-216, 225.

roots of (1) are equal they must be $+1$ or -1 . We obtain then (possibly by a transformation of determinant not unity) the canonical form

$$C: \begin{pmatrix} \pm 1, & \pm 1 \\ 0, & \pm 1 \end{pmatrix}.$$

We define $\kappa = \frac{1}{2}(\alpha + \delta)$ as the invariant* of the substitution $\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$.

If two substitutions of H have the same invariant, they have the same characteristic equation, and therefore the same canonical form. If two substitutions U and V of H have the same canonical form, there exists† a binary linear homogeneous substitution W belonging to the $GF[p^n]$ (but not necessarily of determinant unity) such that $U = W^{-1}VW$. Precisely as in §225 (Dickson, *Linear Groups*) we can prove that if U and V have the same canonical form A or B , there exists a substitution W_1 of H which transforms U into V ; also that every substitution of H of invariant ± 1 (except $\begin{pmatrix} -1, & 0 \\ 0, & -1 \end{pmatrix}$ and the identity) is conjugate within H to one or other of the types

$$C_0: \begin{pmatrix} \pm 1, & \pm 1 \\ 0, & \pm 1 \end{pmatrix},$$

$$C_1: \begin{pmatrix} \pm 1, & \pm \mu \\ 0, & \pm 1 \end{pmatrix},$$

where μ is a particular not-square in the $GF[p^n]$; and further that the two types C_0, C_1 are not conjugate within H . Hence we have the result:

Two substitutions of H having the same invariant (not ± 1) are conjugate within H .

A) Let ρ be a primitive root of the $GF[p^n]$. The substitution $R = \begin{pmatrix} \rho, & 0 \\ 0, & \rho^{-1} \end{pmatrix}$ is of period $s - 1$. To study the conjugacy of the substitutions $R^a = \begin{pmatrix} \rho^a, & 0 \\ 0, & \rho^{-a} \end{pmatrix}$ we transform R^a by $U = \begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$, $\alpha\delta - \beta\gamma = 1$, and obtain

$$V = U^{-1}R^aU = \begin{pmatrix} \alpha\delta\rho^a - \beta\gamma\rho^{-a}, & -\alpha\beta(\rho^a - \rho^{-a}) \\ \gamma\delta(\rho^a - \rho^{-a}), & -\beta\gamma\rho^a + \alpha\delta\rho^{-a} \end{pmatrix}.$$

In order that V shall be identical with R^a (i. e., U commutative with R^a) it is

*In the case of F' we define $\kappa = \pm \frac{1}{2}(\alpha + \delta)$ as the invariant of the substitution $\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$.

† Dickson, *Linear Groups*, § 216.

necessary that either $\alpha\beta = \gamma\delta = 0$, or $\rho^a = \rho^{-a} = 0$. The first alternative leads to two cases:

- 1) if $\beta = \gamma = 0$ then $V = \begin{pmatrix} \rho^a & 0 \\ 0 & \rho^{-a} \end{pmatrix} = R^a$, $U = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$;
- 2) if $\alpha = \delta = 0$ then $V = \begin{pmatrix} \rho^{-a} & 0 \\ 0 & \rho^{-a} \end{pmatrix} = R^{-a}$, $U = \begin{pmatrix} 0 & \beta \\ -\beta^{-1} & 0 \end{pmatrix}$.

If $R^a \neq R^{-a}$ we have $s-1$ substitutions U commutative with R^a ; and therefore R^a is one of $s(s^2-1) \div (s-1) = s(s+1)$ conjugate substitutions. If $R^a = R^{-a}$ then $\rho^a = \rho^{-a}$, which is the second alternative. According as $\rho^a = +1$ or -1 , R^a is the identity or $R^{\frac{s-1}{2}} \equiv \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$; each of these substitutions is conjugate only to itself. With the exception of these the powers of R are conjugate in pairs, thus representing $\frac{1}{2}(s-3)$ classes of conjugate substitutions, each class containing $s(s+1)$ substitutions.*

B) The group H is holodrically isomorphic with the group† $G_{2,p,n}$ of substitutions

$$U = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}, \quad (A\bar{A} + B\bar{B} = 1),$$

where $\bar{A} \equiv A^s$ is the conjugate of A with respect to the $GF[p^n]$. If σ is a primitive root of the equation $\sigma^{s+1} = 1$, so that $\bar{\sigma} = \sigma^{-1}$, then the substitution $S = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix}$ is of period $s+1$. As above we find that the powers of S , except $S^{\frac{s+1}{2}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and S^{s+1} , which is the identity, are conjugate in pairs, viz., S^b with S^{-b} . There are $\frac{1}{2}(s-1)$ classes represented by the powers of S , each containing $s(s-1)$ substitutions.‡

*In the case of the group H' if $R^a = R^{-a}$ either $\rho^a = \rho^{-a}$, i. e., R^a is the identity, or $\rho^a = -\rho^{-a}$, which is possible only if $s-1$ is divisible by 4; $R^a \equiv R^{\frac{s-1}{4}}$ is commutative with $s-1$ substitutions U , and is therefore one of $\frac{1}{2}s(s+1)$ conjugate substitutions. If we define ε as $+1$ or -1 according as s has the form $4l+1$ or $4l-1$, where l is an integer, then we have $\frac{1}{4}(s-2+\varepsilon)$ classes represented by the powers of R , each containing $(ss+1)$ substitutions except the class of period two, which contains $\frac{1}{2}s(s+1)$ substitutions.

†Dickson, *Linear Groups*, p. 132.

‡The substitution $S = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix}$ of H' is of period $\frac{1}{2}(s+1)$; the substitutions S^b (not the identity) are conjugate in pairs except when $\varepsilon = -1$, and then $S^{\frac{s+1}{4}}$ is conjugate only to itself and is of period two. We have $\frac{1}{4}(s-\varepsilon)$ classes, each containing $s(s-1)$ substitutions, except the class of period two, which contains $\frac{1}{2}s(s-1)$ substitutions.

The numbers $\pm a$ ($\pm b$) taken mod. $s - 1$ (mod. $s + 1$) will be called indifferently the index of the class represented by $R^a(S^b)$. We have defined $\kappa = \frac{1}{2}(\alpha + \delta)$ as the invariant of the substitution $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. The substitutions* $R^a(S^b)$ are characterized by the property that $\kappa^2 - 1$ is a square (not-square) in the $GF[p^n]$.

C) Consider the substitution

$$T_\mu = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}, \quad \mu \text{ a mark } \neq 0 \text{ of the } GF[p^n].$$

Transforming T_μ by $U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, $\alpha\delta - \beta\gamma = 1$, we obtain

$$V = U^{-1} T_\mu U = \begin{pmatrix} 1 - \alpha\gamma\mu & \alpha^2\mu \\ -\gamma^2\mu & 1 + \alpha\gamma\mu \end{pmatrix}.$$

T_μ is commutative only with those substitutions U for which $\gamma = 0$, $\alpha = \pm 1$, in number $2s$; hence T_μ is one of $s(s^2 - 1) \div 2s = \frac{1}{2}(s^2 - 1)$ conjugate substitutions. We observe that the conjugate substitutions T_μ and V have the property that μ and $\alpha^2\mu$ ($\alpha \neq 0$), or μ and $\gamma^2\mu$ in case $\alpha = 0$, are both squares or both not-squares in the $GF[p^n]$. This condition can easily be proved to be sufficient for the conjugacy of T_μ and V ; i. e., a substitution $Q = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$, $\alpha'\delta' - \beta'\gamma' = 1$, of invariant unity is conjugate to T_μ if μ and β' ($\neq 0$), or μ and $-\gamma'$ in case $\beta' = 0$, are both squares or both not-squares in the $GF[p^n]$.

The substitutions $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ (except the identity) of invariant $+1$ will be said to belong to the class (μ) or to the class (ν) according as β ($\neq 0$), or $-\gamma$ if $\beta = 0$, is a square or a not-square in the $GF[p^n]$. Similarly the substitutions $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, except $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, of invariant -1 will be said to belong to the class† (m) or to the class (n) according as β ($\neq 0$), or $-\gamma$ if $\beta = 0$, is a square or a not-square in the $GF[p^n]$.

* The substitutions of period two have the invariant zero.

† To the two classes (μ) and (m) of H corresponds the one class (μ) of F ; and similarly for (ν) and (n) .

The substitutions* $\begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$ and $\begin{pmatrix} -1, & 0 \\ 0, & -1 \end{pmatrix}$ will be denoted by (λ) and (l) respectively.

The total number † of classes of conjugate substitutions is $\frac{1}{2}(s-3) + \frac{1}{2}(s-1) + 4 + 2 = s + 4$.

Since the powers of R and S were shown to be conjugate with their reciprocals, a substitution W and its reciprocal W^{-1} belong to the same class except when W belongs to one of the classes (μ) , (ν) , (m) , (n) .

§ 2.

We define ε_κ to be $+1$ or -1 according as $\kappa^2 - 1$ is a square or a not-square in the $GF[p^n]$, where κ denotes henceforth any invariant except ± 1 . In place of ε_0 we write $\varepsilon_\dagger$.

The class whose invariant is κ is denoted by (κ) . This notation is unique. For, two conjugate substitutions have the same invariant; and it has been proved that if two substitutions of H have the same invariant (not ± 1) they are conjugate. Instead of κ we shall nearly always use $\alpha, \beta, \gamma, \dots$, and we shall denote the indices of the classes $(\alpha), (\beta), (\gamma), \dots$ by $\pm a, \pm b, \pm c, \dots$ respectively. These indices are taken mod. $s-1$ or mod. $s+1$ according as $\kappa^2 - 1$ is a square or a not-square in the $GF[p^n]$.

Denoting by h_θ the number of substitutions in a class (θ) we have §

$$h_\lambda = h_l = 1, \quad h_\mu = h_\nu = h_m = h_n = \frac{1}{2}(s^2 - 1), \quad h_\kappa = s(s + \varepsilon_\kappa).$$

If $\varepsilon_\alpha = \varepsilon$, $\varepsilon_\beta = -\varepsilon$, the numbers || of classes (α) and (β) are $\frac{1}{2}(s-2-\varepsilon)$ and $\frac{1}{2}(s-2+\varepsilon)$ respectively.

We define ¶ ζ_κ as $+1$ or -1 according as $-2(1-\kappa)$ is a square or a not-square in the $GF[p^n]$. Then $\zeta_\alpha = \varepsilon_\alpha(-1)^\alpha$.

* The one corresponding substitution $\begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$ of F will be denoted by (λ) .

† For F the total number of classes is $\frac{1}{2}(s+5)$.

‡ For the definition of ε compare p. 390, foot-note*.

§ For F we have $h_\lambda = 1$, $h_\mu = h_\nu = \frac{1}{2}(s^2 - 1)$, $h_\kappa = s(s + \varepsilon_\kappa)$, $h_0 = \frac{1}{2}s(s + \varepsilon)$. The class (0) requires to be distinguished from the other classes (κ) more frequently in the case of F than in the case of H .

|| For F the numbers of classes (α) and (β) are $\frac{1}{4}(s-\varepsilon)$ and $\frac{1}{4}(s-2+\varepsilon)$ respectively.

¶ In the case of F we define η_κ as $+1, -1$, or 0 according as $-2(1+\kappa)$ and $-2(1-\kappa)$ are both squares, both not-squares, or one a square and the other a not-square, in the $GF[p^n]$. We also define $2\eta_1 = \varepsilon$. If $\varepsilon\alpha = \varepsilon$ then $\eta_\alpha = (-1)^\alpha\varepsilon$; if $\varepsilon\beta = -\varepsilon$ then $\eta_\beta = 0$; further, $\eta_\kappa = \frac{1}{2}(1 + \varepsilon\varepsilon_\kappa)\zeta_\kappa$.

§ 3.

Three (distinct or equal) classes (α) , (β) , (γ) are called *concordant* if between their invariants there exists the relation*

$$\alpha^2 + \beta^2 + \gamma^2 - 2\alpha\beta\gamma = 1; \quad (1)$$

otherwise they are called *discordant*. If we write (1) in the form

$$(\alpha^2 - 1)(\beta^2 - 1) = (\alpha\beta - \gamma)^2$$

it follows that $\varepsilon_\alpha = \varepsilon_\beta$; similarly $\varepsilon_\beta = \varepsilon_\gamma$. Hence three concordant classes must all be represented by powers of R or all by powers of S . If α and β are given we find that the classes whose invariants are

$$\gamma = \alpha\beta + \sqrt{(\alpha^2 - 1)(\beta^2 - 1)}, \quad \delta = \alpha\beta - \sqrt{(\alpha^2 - 1)(\beta^2 - 1)} \quad (2)$$

are concordant with (α) and (β) . If $\beta = \alpha$ then $\gamma = 2\alpha^2 - 1$; and therefore we have $\varepsilon_\alpha = \varepsilon_{2\alpha^2 - 1}$.

Let r denote ρ or σ according as $\varepsilon_\alpha = +1$ or -1 . Substituting the values $2\alpha = r^a + r^{-a}$, etc., in (1), and factoring, we obtain

$$(r^{a+b+c} - 1)(r^{-a-b+c} - 1)(r^{-a+b-c} - 1)(r^{a-b-c} - 1) = 0.$$

Hence $a \pm b \pm c \equiv 0 \pmod{s-1}$ or $s+1$ according as $\varepsilon_\alpha = +1$ or -1 . The indices of the two classes (γ) and (δ) concordant with (α) and (β) are therefore

$$c \equiv a + b, \quad d \equiv a - b \pmod{s-1, s+1 \text{ respectively}}.$$

§ 4.

Let Θ , Φ , Ψ represent the substitutions of any three distinct or equal classes (θ) , (ϕ) , (ψ) respectively, and let (θ') , (ϕ') , (ψ') denote the classes of the inverse substitutions Θ^{-1} , Φ^{-1} , Ψ^{-1} respectively. If Θ , Φ , Ψ run through all the substitutions of their respective classes, we denote† by $h_{\theta\phi\psi}$ the number of times we obtain the relation $\Theta\Phi\Psi = E$ (the identity), or $\Theta\Phi = \Psi^{-1}$. The subscripts θ , ϕ , ψ may be permuted in any manner.

To obtain $h_{\alpha\beta\gamma}$ we determine $\frac{h_{\alpha\beta\gamma}}{h_\alpha}$; we take a *particular* substitution of (α) , compound it with all the substitutions of (β) ,

$$\begin{pmatrix} \alpha, & 1 \\ \alpha^2 - 1, & \alpha \end{pmatrix} \begin{pmatrix} \xi, \eta \\ \zeta, 2\beta - \xi \end{pmatrix} = \begin{pmatrix} \alpha\xi + \eta(\alpha^2 - 1), & \xi + \alpha\eta \\ \alpha\zeta + (2\beta - \xi)(\alpha^2 - 1), & \zeta + \alpha(2\beta - \xi) \end{pmatrix},$$

and determine how many of the resulting substitutions belong to the class (γ') .

* For F this relation takes the form $\alpha^2 + \beta^2 + \gamma^2 \pm 2\alpha\beta\gamma = 1$.

† Frobenius, *Über Gruppencharaktere*, 1896, pp. 987, 988.

In order that the resulting substitutions may be of class $(\gamma') \equiv (\gamma)^*$ and have the determinant unity, we must have

$$\begin{aligned}\alpha\xi + \eta(\alpha^2 - 1) + \zeta + \alpha(2\beta - \xi) &= 2\gamma, \\ \xi(2\gamma - \xi) - \eta\zeta &= 1.\end{aligned}$$

The number of distinct sets of solutions ξ, η, ζ of these equations will give $\frac{h_{\alpha\beta\gamma}}{h_\alpha}$. Eliminating ζ we obtain

$$(\xi - \beta)^2 - (\alpha^2 - 1)\left(\eta + \frac{\alpha\beta - \gamma}{\alpha^2 - 1}\right)^2 = \frac{(\alpha^2 - 1)(\beta^2 - 1) - (\alpha\beta - \gamma)^2}{\alpha^2 - 1}.$$

If $(\alpha), (\beta), (\gamma)$ are discordant the right-hand side is distinct from zero, and we obtain $s - \varepsilon_\alpha$ sets of solutions†. If $(\alpha), (\beta), (\gamma)$ are concordant‡ the right-hand side is zero, and we obtain $s + \varepsilon_\alpha(s - 1)$ sets of solutions. Hence $h_{\alpha\beta\gamma} = h + \varepsilon_\alpha s^2(s + \varepsilon_\alpha)$ or h according as $(\alpha), (\beta), (\gamma)$ are concordant or discordant.

If we denote the substitutions of (μ) and (ν) by P and Q respectively, then according as $\varepsilon = 1$ or -1 will P^{-1} belong to (μ) or (ν) , and Q^{-1} to (ν) or (μ) . Hence $h_{\lambda\mu\nu} = \frac{1}{2}h_\mu(1 - \varepsilon)$. Similarly $h_{\lambda mn} = \frac{1}{2}h_\mu(1 - \varepsilon)$.

The group H is self-conjugate under the group of all binary linear homogeneous substitutions of determinant $\neq 0$; by a substitution of determinant a not-square in the $GF[p^n]$ the class (μ) is transformed into (ν) , and (ν) into (μ) , and simultaneously (m) into (n) and (n) into (m) . Hence the notations (μ) and (ν) are interchangeable; likewise (m) and (n) ; furthermore the interchange of (μ) and (ν) must be accompanied by the interchange of (m) and (n) , and vice versa.

To determine $h_{\mu\mu\nu}$ we compute $h_\mu(h_{\mu\nu\mu} + h_{\mu\nu\nu} + h_{\mu\nu\lambda})$; we take a definite substitution of (μ) , compound it with all the substitutions of (ν) ,

$$\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} \xi & \eta \\ \zeta & 2 - \xi \end{pmatrix} = \begin{pmatrix} -\eta & \xi + 2\eta \\ -2 + \xi & \zeta + 4 - 2\xi \end{pmatrix},$$

* See last paragraph of §1.

† Dickson, *Linear Groups*, p. 46.

‡ In the case of F we have the equations

$$(\xi - \beta)^2 - (\alpha^2 - 1)\left(\eta + \frac{\alpha\beta \pm \gamma}{\alpha^2 - 1}\right)^2 = \frac{(\alpha^2 - 1)(\beta^2 - 1) - (\alpha\beta \pm \gamma)^2}{\alpha^2 - 1}.$$

If $(\alpha), (\beta), (\gamma)$ are concordant then one of the relations $(\alpha^2 - 1)(\beta^2 - 1) - (\alpha\beta \pm \gamma)^2 = 0$ holds and not the other. Suppose that $(\alpha^2 - 1)(\beta^2 - 1) - (\alpha\beta + \gamma)^2 = 0$; then the equations with the upper and lower signs have $s + \varepsilon_\alpha(s - 1)$ and $s - \varepsilon_\alpha$ sets of solutions respectively; in all $2(s - \varepsilon_\alpha) + \varepsilon_\alpha s$ sets of solutions.

and find how many of the resulting substitutions belong to the classes (λ) , (μ) , (ν) collectively, i. e., have the invariant $+1$. We have therefore the equations

$$\begin{aligned}\xi(2 - \xi) - \eta\zeta &= 1, \\ -\eta + \zeta + 4 - 2\xi &= 2.\end{aligned}$$

Eliminating ξ we get $(\eta + \zeta)^2 = 0$, or $-\zeta = \eta$. We can take for η every not-square in the $GF[p^n]$, thus obtaining $\frac{1}{2}(s-1)$ sets of solutions. But $h_{\mu\nu\mu} = h_{\nu\mu\nu} = h_{\mu\nu\nu}$; hence $h_{\mu\mu\nu} = \frac{1}{4}h_\mu(s-2+\varepsilon)$.

Proceeding in this way we get the following results.*

$$\begin{aligned}h_{\alpha\beta\gamma} &= h + \varepsilon_\alpha s^2(s + \varepsilon_\alpha) \text{ or } h \text{ according as } (\alpha), (\beta), (\gamma) \text{ are concordant or discordant,} \\ h_{\alpha\alpha\lambda} &= h_\alpha, h_{\alpha\alpha l} = 0, h_{\alpha\alpha\mu} = h_{\alpha\alpha\nu} = \frac{1}{2}h(1 + \varepsilon_\alpha), h_{\alpha\alpha m} = h_{\alpha\alpha n} = \frac{1}{2}h \text{ if } \alpha \neq 0, \\ h_{\alpha\alpha m} &= h_{\alpha\alpha n} = \frac{1}{2}h(1 + \varepsilon), h_{-\alpha\alpha l} = h_\alpha, h_{-\alpha\alpha m} = \frac{1}{2}h(1 + \varepsilon_\alpha), \\ h_{\alpha\beta\lambda} &= 0, h_{\alpha\beta\mu} = h_{\alpha\beta\nu} = h_{\alpha\beta m} = h_{\alpha\beta n} = \frac{1}{2}h, \\ h_{\alpha\lambda\lambda} &= h_{\alpha\lambda\mu} = h_{\alpha\lambda\nu} = h_{\alpha\lambda m} = h_{\alpha\lambda n} = h_{\alpha l\mu} = h_{\alpha l\nu} = h_{\alpha l m} = h_{\alpha l n} = 0, \\ h_{\alpha\mu\mu} &= h_{\alpha\nu\nu} = h_{\alpha m m} = h_{\alpha n n} = \frac{1}{4}h(1 + \varepsilon\zeta_\alpha), h_{\alpha\mu\nu} = h_{\alpha m n} = \frac{1}{4}h(1 - \varepsilon\zeta_\alpha), \\ h_{\alpha\mu m} &= h_{\alpha\nu n} = \frac{1}{4}h(1 + \varepsilon_\alpha\zeta_\alpha), h_{\alpha m \nu} = h_{\alpha\mu n} = \frac{1}{4}h(1 - \varepsilon_\alpha\zeta_\alpha), \\ h_{l\mu m} &= h_{l\nu n} = h_{\lambda\mu\mu} = h_{\lambda\nu\nu} = h_{\lambda m m} = h_{\lambda n n} = \frac{1}{2}h_\mu(1 + \varepsilon), \\ h_{\lambda m \nu} &= h_{\lambda\mu m} = h_{\lambda\mu\nu} = h_{\lambda\nu m} = h_{l\mu\mu} = h_{l\nu\nu} = h_{l\mu\nu} = h_{l m n} = 0, \\ h_{l m m} &= h_{l n n} = h_{\mu\mu n} = h_{m\nu\nu} = h_{\mu m \nu} = h_{\mu\nu n} = 0, \\ h_{m m n} &= h_{m n n} = 0, h_{\lambda\mu\nu} = h_{\lambda m n} = h_{l\mu n} = h_{l m \nu} = \frac{1}{2}h_\mu(1 - \varepsilon), \\ h_{\mu\mu\mu} &= h_{\nu\nu\nu} = h_{\mu m m} = h_{\nu n n} = \frac{1}{4}h_\mu(s - 2 - 3\varepsilon), \\ h_{\mu\mu\nu} &= h_{\mu\nu\nu} = h_{\mu n n} = h_{m m \nu} = h_{\mu m n} = h_{m \nu n} = \frac{1}{4}h_\mu(s - 2 + \varepsilon), \\ h_{\mu\mu m} &= h_{\nu\nu n} = h_{m m m} = h_{n n n} = s h_\mu.\end{aligned}$$

§ 5.

The value of a group-character χ for any class (θ) is denoted by χ_θ ; but sometimes the value of a group-character for a class represented by a power of R or S is denoted by $\chi(R^a)$ or $\chi(S^b)$ respectively. Also $\chi_\lambda = f$.

* These results reduce to a very few in the case of F , since $(l) = (\lambda)$, $(m) = (\mu)$, $(n) = (\nu)$. For F we have the following additional results: $h_{\alpha\alpha\beta} = 2h + \varepsilon s^2(s + \varepsilon)$ or $2h$ according as (α) , (β) are concordant or discordant; $h_{\alpha\alpha\alpha} = \frac{1}{2}h$, $h_{\alpha\alpha\alpha} = h$, $h_{\alpha\alpha\lambda} = \frac{1}{2}s(s + \varepsilon)$, $h_{\alpha\alpha\mu} = h_{\alpha\alpha\nu} = \frac{1}{2}h(1 + \varepsilon)$, $h_{\alpha\alpha m} = h_{\alpha\alpha n} = h$, $h_{\alpha\mu\mu} = h_{\alpha\nu\nu} = \frac{1}{2}h(1 + \varepsilon\eta)$, $h_{\alpha\mu\nu} = \frac{1}{2}h(1 - \varepsilon\eta)$, where $\eta = \eta_\alpha$.

We make the following abbreviations:*

$$\begin{aligned}x &= \frac{1}{2} (\chi_\mu + \chi_\nu + \chi_m + \chi_n) + \sum_{\kappa} \chi_{\kappa}, \\y &= \frac{\varepsilon}{4} (\chi_\mu + \chi_\nu + \chi_m + \chi_n) + \sum_{\kappa} \zeta_{\kappa} \chi_{\kappa}, \\z &= \frac{\varepsilon}{4} (\chi_\mu + \chi_\nu + \chi_m + \chi_n) + \sum_{\kappa} \varepsilon_{\kappa} \zeta_{\kappa} \chi_{\kappa}.\end{aligned}$$

From the relation †

$$h_{\theta} h_{\psi} \chi_{\theta} \chi_{\psi'} = f \sum_{\phi} h_{\theta' \phi \psi} \chi_{\phi},$$

where (θ) , (ϕ) , (ψ) are any three classes, we derive the following set of equations ‡

$$\chi_i^2 = f^2, \quad (1)$$

$$\chi_{\alpha} \chi_i = f \chi_{-\alpha}, \quad (2)$$

$$s \chi_{\alpha} \chi_{\beta} = f x, \quad (\varepsilon_{\beta} = -\varepsilon_{\alpha}), \quad (3)$$

$$\frac{s(s + \varepsilon_{\alpha})}{f} \chi_{\alpha} \chi_{\beta} = x(s - \varepsilon_{\alpha}) + \varepsilon_{\alpha} s (\chi_{\gamma} + \chi_{\delta}), \quad (\varepsilon_{\beta} = \varepsilon_{\alpha}, \alpha \neq -\beta), \quad (4)$$

where γ and δ are determined by (3), §3,

$$\frac{s(s + \varepsilon_{\alpha})}{f} \chi_{\alpha} \chi_{-\alpha} = \chi_i + x(s - \varepsilon_{\alpha}) + \frac{1}{2} \varepsilon_{\alpha} (s - \varepsilon_{\alpha}) (\chi_m + \chi_n) + \varepsilon_{\alpha} s \chi_{-(2\alpha^2 - 1)}, \quad (5)$$

$$\frac{s(s + \varepsilon_{\alpha})}{f} \chi_{\alpha}^2 = f + x(s - \varepsilon_{\alpha}) + \frac{1}{2} \varepsilon_{\alpha} (s - \varepsilon_{\alpha}) (\chi_{\mu} + \chi_{\nu}) + \varepsilon_{\alpha} s \chi_{2\alpha^2 - 1}, \quad (6)$$

$$\frac{s(s + \varepsilon)}{f} \chi_0^2 = f + x(s - \varepsilon) + \frac{1}{2} \varepsilon (s - \varepsilon) (\chi_{\mu} + \chi_{\nu} + \chi_m + \chi_n), \quad (7)$$

$$\frac{s + \varepsilon_{\alpha}}{f} \chi_{\alpha} \chi_{\mu} = x + \varepsilon_{\alpha} \chi_{\alpha} + \frac{1}{2} \zeta_{\alpha} (\chi_{\mu} - \chi_{\nu}) + \frac{1}{2} \varepsilon \varepsilon_{\alpha} \zeta_{\alpha} (\chi_m - \chi_n), \quad (8)$$

$$\frac{s + \varepsilon_{\alpha}}{f} \chi_{\alpha} \chi_{\nu} = x + \varepsilon_{\alpha} \chi_{\alpha} - \frac{1}{2} \zeta_{\alpha} (\chi_{\mu} - \chi_{\nu}) - \frac{1}{2} \varepsilon \varepsilon_{\alpha} \zeta_{\alpha} (\chi_m - \chi_n), \quad (8a)$$

$$\frac{s + \varepsilon_{\alpha}}{f} \chi_{\alpha} \chi_m = x + \varepsilon_{\alpha} \chi_{-\alpha} + \frac{1}{2} \varepsilon \varepsilon_{\alpha} \zeta_{\alpha} (\chi_{\mu} - \chi_{\nu}) + \frac{1}{2} \zeta_{\alpha} (\chi_m - \chi_n), \quad (9)$$

$$\frac{s + \varepsilon_{\alpha}}{f} \chi_{\alpha} \chi_n = x + \varepsilon_{\alpha} \chi_{-\alpha} - \frac{1}{2} \varepsilon \varepsilon_{\alpha} \zeta_{\alpha} (\chi_{\mu} - \chi_{\nu}) - \frac{1}{2} \zeta_{\alpha} (\chi_m - \chi_n), \quad (9a)$$

* In the case of F we make the abbreviations

$$\begin{aligned}x &= \chi_0 + \chi_{\mu} + \chi_{\nu} + 2 \sum_{\kappa} \chi_{\kappa}, \\y &= \eta_0 \chi_0 + \eta_1 (\chi_{\mu} + \chi_{\nu}) + 2 \sum_{\kappa} \eta_{\kappa} \chi_{\kappa}.\end{aligned} \quad \kappa \neq 0,$$

† Über Gruppencharaktere, p. 994.

‡ We get the equations for F from (3), (4), (6)–(8a), (10)–(10b) if we set $\chi_m = \chi_{\mu}$, $\chi_n = \chi_{\nu}$, and remember that $\frac{1}{2} (1 + \varepsilon \varepsilon_{\alpha}) \zeta_{\alpha} = \eta_{\alpha}$.

$$\begin{aligned} \frac{s^2-1}{fs} \chi_\mu^2 &= x + \varepsilon y + (1+\varepsilon) \frac{f}{s} - \frac{s+2\varepsilon+4}{4s} (\chi_\mu + \chi_\nu) \\ &\quad - \frac{1}{s} (\chi_\mu - \chi_\nu) + \frac{1}{4} (\chi_m + \chi_n) + \varepsilon (\chi_m - \chi_n), \end{aligned} \quad (10)$$

$$\begin{aligned} \frac{s^2-1}{fs} \chi_\nu^2 &= x + \varepsilon y + (1+\varepsilon) \frac{f}{s} - \frac{s+2\varepsilon+4}{4s} (\chi_\mu + \chi_\nu) + \frac{1}{s} (\chi_\mu - \chi_\nu) \\ &\quad + \frac{1}{4} (\chi_m + \chi_n) - \varepsilon (\chi_m - \chi_n), \end{aligned} \quad (10a)$$

$$\frac{s^2-1}{fs} \chi_\mu \chi_\nu = x - \varepsilon y + (1-\varepsilon) \frac{f}{s} + \frac{s+2\varepsilon-4}{4s} (\chi_\mu + \chi_\nu) - \frac{1}{4} (\chi_m + \chi_n), \quad (10b)$$

$$\begin{aligned} \frac{s^2-1}{fs} \chi_\mu \chi_m &= x + z + \frac{1+\varepsilon}{s} \chi_l + \frac{2-\varepsilon}{4} (\chi_\mu + \chi_\nu) + \varepsilon (\chi_\mu - \chi_\nu) \\ &\quad - \frac{\varepsilon s + 2\varepsilon + 4}{4s} (\chi_m + \chi_n) - \frac{1}{s} (\chi_m - \chi_n), \end{aligned} \quad (11)$$

$$\begin{aligned} \frac{s^2-1}{fs} \chi_\nu \chi_n &= x + z + \frac{1+\varepsilon}{s} \chi_l + \frac{2-\varepsilon}{4} (\chi_\mu + \chi_\nu) - \varepsilon (\chi_\mu - \chi_\nu) \\ &\quad - \frac{\varepsilon s + 2\varepsilon + 4}{4s} (\chi_m + \chi_n) + \frac{1}{s} (\chi_m - \chi_n), \end{aligned} \quad (11a)$$

$$\begin{aligned} \frac{s^2-1}{fs} \chi_\mu \chi_n &= \frac{s^2-1}{fs} \chi_m \chi_\nu = x - z + \frac{1-\varepsilon}{s} \chi_l \\ &\quad - \frac{2-\varepsilon}{4} (\chi_\mu + \chi_\nu) + \frac{\varepsilon s + 2\varepsilon - 4}{4s} (\chi_m + \chi_n), \end{aligned} \quad (11b)$$

$$\chi_l \chi_\mu = f \chi_m, \quad (12)$$

$$\chi_l \chi_\nu = f \chi_n. \quad (12a)$$

I. We seek first those solutions for which χ_μ and χ_ν are distinct. Then, according to (12) and (12a), χ_m and χ_n are distinct. From (1) we find $\chi_l = \pm f$.

Suppose first that $\chi_l = f$. Then $\chi_m = \chi_\mu$ and $\chi_n = \chi_\nu$, and from (8) and (8a) we obtain $\frac{s+\varepsilon\alpha}{f} \chi_a = \zeta_a (1 + \varepsilon \varepsilon_a)$. If $\varepsilon_a = -\varepsilon$ then $\chi_a = 0$; if $\varepsilon_a = \varepsilon$, and if we set the proportionality factor* $f = \frac{1}{2}(s + \varepsilon)$, we obtain $\chi_a = \zeta_a$. Hence $\chi(R^a) = \frac{(1+\varepsilon)(-1)^a}{2}$, $\chi(S^b) = -\frac{(1-\varepsilon)(-1)^b}{2}$. According to (3), $x = 0$. If in (6) $\varepsilon_a = -\varepsilon$, then $\chi_\mu + \chi_\nu = \varepsilon$, and $y = \frac{1}{2} + \sum_{\kappa} \zeta_{\kappa}^2$, where $\varepsilon_{\kappa} = \varepsilon$. The number of classes (κ) for which $\varepsilon_{\kappa} = \varepsilon$ is $\frac{1}{2}(s - 2 - \varepsilon)$, and therefore $y = \frac{1}{2}(s - 1 - \varepsilon)$. Similarly $z = \frac{\varepsilon}{2}(s - 2)$. From (10b) we get $4\chi_\mu \chi_\nu = 1 - \varepsilon s$;

* The proportionality factor may be chosen arbitrarily. See *Über Gruppencharaktere*, p. 999.

also we already have $\chi_\mu + \chi_\nu = \varepsilon$. Hence $\chi_\mu = \chi_m = \frac{1}{2}(\varepsilon \pm \sqrt{\varepsilon s})$, $\chi_\nu = \chi_n = \frac{1}{2}(\varepsilon \mp \sqrt{\varepsilon s})$. These values of the characters will be found to satisfy all the equations.

Let next $\chi_l = -f$. Then $\chi_m = -\chi_\mu$, $\chi_n = -\chi_\nu$. From (8) and (8a) we get as before $\frac{s + \varepsilon_a}{f} \chi_a = \zeta_a (1 - \varepsilon \varepsilon_a)$. If $\varepsilon_a = \varepsilon$, $\chi_a = 0$; if $\varepsilon_a = -\varepsilon$, and if we set $f = \frac{1}{2}(s - \varepsilon)$, we obtain* $\chi_a = \zeta_a$. According to (3), $x = 0$; if in (6) $\varepsilon_a = \varepsilon$ then $\chi_\mu + \chi_\nu = -\varepsilon$, and therefore $\chi_m + \chi_n = \varepsilon$. Also $y = \frac{1}{2}(s - 2 + \varepsilon)$ and $z = -\frac{1}{2}(s - 2 + \varepsilon)$. From (10b) we obtain $4\chi_\mu \chi_\nu = 1 - \varepsilon s$; this combined with $\chi_\mu + \chi_\nu = -\varepsilon$ gives $\chi_\mu = -\chi_m = \frac{1}{2}(-\varepsilon \pm \sqrt{\varepsilon s})$, $\chi_\nu = -\chi_n = \frac{1}{2}(-\varepsilon \mp \sqrt{\varepsilon s})$.

II. For all other solutions $\chi_\mu = \chi_\nu$, and therefore $\chi_m = \chi_n$. Instead of equations (10)–(11b) we shall use the following which are obtained from them by addition and subtraction:

$$\frac{s^2 - 1}{f} \chi_\mu^2 = sx + f - 2\chi_\mu, \quad (10')$$

$$2\varepsilon s y = (s + 2\varepsilon)\chi_\mu - s\chi_m - 2\varepsilon f, \quad (10'')$$

$$\frac{s^2 - 1}{f} \chi_\mu \chi_m = sx + \chi_l - 2\chi_m, \quad (11')$$

$$2\varepsilon s z = s(1 - 2\varepsilon)\chi_\mu + (s + 2)\chi_m - 2\chi_l. \quad (11'')$$

We seek first those solutions for which x is distinct from zero. According to (3) none of the characters χ_a can be zero; and $\chi_a = \chi_\nu$ if $\varepsilon_a = \varepsilon_\nu$, i. e., all characters χ_a are equal for which ε_a has the same sign. Since $\chi_a = \chi_{-a}$ it follows from (8) and (9) that $\chi_m = \chi_\mu$, and therefore from (12) that $\chi_l = f$.

Let $\varepsilon_a = \varepsilon$, $\varepsilon_\beta = -\varepsilon$. Then* $y = \varepsilon + \chi_a \sum_a \zeta_a + \chi_\beta \sum_\beta \zeta_\beta = \varepsilon - \varepsilon \chi_a$. From (6) and (8) we obtain

$$\begin{aligned} \chi_a &= \frac{1}{s} \{ (s - \varepsilon) \chi_\mu + \varepsilon f \}, \\ \chi_\beta &= \frac{1}{s} \{ (s + \varepsilon) \chi_\mu - \varepsilon f \}. \end{aligned} \quad (A)$$

In the sum $x = \frac{1}{2}(\chi_\mu + \chi_\nu + \chi_m + \chi_n) + \sum_{a,\beta} (\chi_a + \chi_\beta)$, $\varepsilon_a = \varepsilon$, $\varepsilon_\beta = -\varepsilon$, the numbers of characters χ_a , χ_β are $\frac{1}{2}(s - 2 - \varepsilon)$, $\frac{1}{2}(s - 2 + \varepsilon)$ respectively. Hence we have

$$x = 2\chi_\mu + \frac{1}{2}(s - 2 - \varepsilon)\chi_a + \frac{1}{2}(s - 2 + \varepsilon)\chi_\beta. \quad (B)$$

* From the definition of x we get $\sum_k \zeta_k = -\varepsilon$ or 0 according as $\varepsilon_k = +\varepsilon$ or $-\varepsilon$. Hence we have in general $\sum_k \zeta_k = -\varepsilon$.

Eliminating χ_a and χ_β from (A) and (B) we obtain

$$sx = (s^2 + 1)\chi_\mu - f.$$

Substituting this value of sx in (10') we obtain

$$\chi_\mu(\chi_\mu - f) = 0.$$

If first $\chi_\mu = 0$, let $f = s$; then $x = y = -1$, $\chi_a = \varepsilon$, $\chi_\beta = -\varepsilon$, in general $\chi_\kappa = \varepsilon_\kappa$, and therefore $\chi(R^a) = 1$, $\chi(S^b) = -1$. Also $\chi_l = s$.

If secondly $\chi_\mu = f$, let $f = 1$; then $\chi_l = \chi_\mu = \chi_m = \chi_a = \chi_\beta = 1$. Also $x = s$, $y = 0$.

III. For all other solutions $x = 0$. Then (3) becomes $\chi_a \chi_\beta = 0$ $\varepsilon_\beta = -\varepsilon_a$. Not all the characters χ_κ can be zero. For, if they were, by giving to ε_a in (6) the values 1 and -1 in turn we would have $\chi_\mu + \chi_\nu = 0$, and therefore $f = 0$, which is inadmissible.

According to (3) either all $\chi_a = 0$ for which $\varepsilon_a = 1$, or all for which $\varepsilon_a = -1$. Suppose first that $\chi_a = 0$ in case $\varepsilon_a = -1$; and let $\chi_l = f = s + 1$. Then, since not all the characters χ_a for $\varepsilon_a = 1$ can be zero, we obtain from (8) $\chi_\mu = 1$; and from (12) $\chi_m = 1$. If $\varepsilon_a = \varepsilon_\beta = 1$, and therefore $\varepsilon_\gamma = \varepsilon_\delta = 1$, we obtain from (4), (5), (6)

$$\chi_a \chi_\beta = \chi_\gamma + \chi_\delta, \chi_a \chi_{-a} = \chi_{-(2a^2-1)} + 2, \chi_a^2 = \chi_{2a^2-1} + 2.$$

If we set $\chi_a = \xi_a$, $\xi_0 = \xi_{\frac{s-1}{2}} = 2$, these equations can be combined into one:

$$\xi_a \xi_b = \xi_{a+b} + \xi_{a-b},$$

where a and b may be distinct or equal. Let r be a new unknown; if we set $\xi_1 = r + r^{-1}$ it follows from $\xi_1 \xi_1 = \xi_2 + \xi_0$ that $\xi_2 = r^2 + r^{-2}$; then from $\xi_1 \xi_2 = \xi_3 + \xi_1$ it follows that $\xi_3 = r^3 + r^{-3}$; in general $\xi_a = r^a + r^{-a}$. From $\xi_{\frac{s-1}{2}} = 2$ we get $r^{\frac{s-1}{2}} = 1$. We obtain then the solutions

$$\begin{aligned} \chi_l = f = s + 1, \quad \chi_\mu = \chi_\nu = \chi_m = \chi_n = 1, \\ \chi_a = r^a + r^{-a} \text{ if } \varepsilon_a = 1, \quad \chi_\beta = 0 \text{ if } \varepsilon_\beta = -1. \end{aligned}$$

From (10''), (11'') we find $y = z = -1$. The above solutions satisfy the equation $x = 0$ except when $r = 1$, and the equations $y = z = -1$ except when $r = -1$; and r can be -1 only when $\varepsilon = 1$. If $\varepsilon = 1$ the equation $r^{\frac{s-1}{2}} = 1$ has $\frac{s-5}{2}$ solutions distinct from ± 1 ; if $\varepsilon = -1$ it has $\frac{s-3}{2}$ solutions distinct from 1; in general it has $\frac{1}{2}(s-4-\varepsilon)$ admissible solutions. Since r^a and r^{-a} give the same value for $r^a + r^{-a}$ these solutions go in pairs, giving $\frac{1}{4}(s-4-\varepsilon)$ characters.

We next let $\chi_l = -f = -(s+1)$. Then $\chi_m = -1$; and (4), (5), (6) become

$$\chi_a \chi_\beta = \chi_\gamma + \chi_\delta, \quad \chi_a \chi_{-a} = \chi_{-(2a^2-1)} - 2, \quad \chi_a^2 = \chi_{2a^2-1} + 2.$$

Setting $\chi_a = \xi_a$, $\xi_0 = 2$, $\xi_{\frac{s-1}{2}} = -2$ we get $\xi_a \xi_b = \xi_{a+b} + \xi_{a-b}$. Let $\xi_1 = r_1 + r_1^{-1}$; then as above we obtain $\xi_a = r_1^a + r_1^{-a}$, and also $r_1^{\frac{s-1}{2}} = -1$. We have the following solutions:

$$f = s+1, \quad \chi_l = -(s+1), \quad \chi_\mu = \chi_\nu = 1, \quad \chi_m = \chi_n = -1, \\ \chi_a = r_1^a + r_1^{-a} \text{ if } \varepsilon_a = 1, \quad \chi_\beta = 0 \text{ if } \varepsilon_\beta = -1.$$

Now $x = 0$, $y = z = \varepsilon - 1$. The above solutions satisfy $x = 0$, and also $y = z = \varepsilon - 1$ except when $r_1 = -1$, which can happen only when $\varepsilon = -1$. These solutions furnish $\frac{1}{4}(s-2+\varepsilon)$ characters.

IV. Suppose finally that $\chi_a = 0$ in case $\varepsilon_a = 1$. Let $f = s-1$. Assuming first that $\chi_l = f$ we get $\chi_m = \chi_\mu = -1$. From (4), (5), (6) we have

$$\chi_a \chi_\beta = -\chi_\gamma - \chi_\delta, \quad \chi_a \chi_{-a} = -\chi_{-(2a^2-1)} + 2, \quad \chi_a^2 = -\chi_{2a^2-1} + 2.$$

Setting $\chi_a = -\xi_a$, $\xi_0 = \xi_{\frac{s+1}{2}} = 2$, we obtain $\xi_a \xi_b = \xi_{a+b} + \xi_{a-b}$. If $\xi_1 = t + t^{-1}$ then $\xi_b = t^b + t^{-b}$, and $t^{\frac{s+1}{2}} = 1$. We have then the following solutions:

$$\chi_l = f = s-1, \quad \chi_\mu = \chi_\nu = \chi_m = \chi_n = -1, \\ \chi_a = 0 \text{ if } \varepsilon_a = 1, \quad \chi_\beta = -(t^b + t^{-b}) \text{ if } \varepsilon_\beta = -1.$$

The equation $x = 0$ is satisfied except when $t = 1$; and the equations $y = -1$, $z = 1 - 2\varepsilon$ are satisfied except when $t = -1$, which can happen only when $\varepsilon = -1$. These solutions furnish $\frac{1}{4}(s-2+\varepsilon)$ characters.

Assuming next that $\chi_l = -f$ we get $\chi_\mu = -1$, $\chi_m = 1$; also

$$\chi_a \chi_\beta = -\chi_\gamma - \chi_\delta, \quad \chi_a \chi_{-a} = -\chi_{-(2a^2-1)} - 2, \quad \chi_a^2 = -\chi_{2a^2-1} + 2.$$

Setting $\chi_a = -\xi_a$, $\xi_0 = 2$, $\xi_{\frac{s+1}{2}} = -2$, we obtain $\xi_a \xi_b = \xi_{a+b} + \xi_{a-b}$. If $\xi_1 = t_1 + t_1^{-1}$, then $\xi_b = t_1^b + t_1^{-b}$, and $t_1^{\frac{s+1}{2}} = 1$. We have the following solutions:

$$f = s-1, \quad \chi_l = -(s-1), \quad \chi_\mu = \chi_\nu = -1, \quad \chi_m = \chi_n = 1, \\ \chi_a = 0 \text{ if } \varepsilon_a = 1, \quad \chi_\beta = -(t_1^b + t_1^{-b}) \text{ if } \varepsilon_\beta = -1.$$

We find that $x = 0$ is satisfied by all these solutions; and that $y = -(1+\varepsilon)$ and $z = 1 + \varepsilon$ are satisfied by all except $t_1 = -1$, which can happen only when $\varepsilon = 1$. These solutions furnish $\frac{1}{4}(s-\varepsilon)$ characters.

The total number of characters thus obtained is

$$4 + 2 + \frac{1}{4}(s-4-\varepsilon) + \frac{1}{4}(s-2+\varepsilon) + \frac{1}{4}(s-2+\varepsilon) + \frac{1}{4}(s-\varepsilon) = s+4.$$

which is equal to the number of classes of conjugate substitutions.

Finally, we readily find that for all these $s+4$ characters the second proportionality factor e is equal to f , where e is defined by

$$\frac{hf}{e} = \sum_{\theta} h_{\theta} \chi_{\theta} \chi_{\theta'}.$$

Below is given a table of the group-characters, N denoting the number of characters in the respective columns.

N	1	1	2	2	$\frac{s-\varepsilon-4}{4}$	$\frac{s+\varepsilon-2}{4}$	$\frac{s+\varepsilon-2}{4}$	$\frac{s-\varepsilon}{4}$
χ_{λ}	1	s	$\frac{s+\varepsilon}{2}$	$\frac{s-\varepsilon}{2}$	$s+1$	$s+1$	$s-1$	$s-1$
χ_{ι}	1	s	$\frac{s+\varepsilon}{2}$	$-\frac{s-\varepsilon}{2}$	$s+1$	$-(s+1)$	$s-1$	$-(s-1)$
χ_{μ}	1	0	$\frac{\varepsilon \pm \sqrt{\varepsilon s}}{2}$	$-\frac{\varepsilon \pm \sqrt{\varepsilon s}}{2}$	1	1	-1	-1
χ_{ν}	1	0	$\frac{\varepsilon \mp \sqrt{\varepsilon s}}{2}$	$-\frac{\varepsilon \mp \sqrt{\varepsilon s}}{2}$	1	1	-1	-1
χ_m	1	0	$\frac{\varepsilon \pm \sqrt{\varepsilon s}}{2}$	$\frac{\varepsilon \mp \sqrt{\varepsilon s}}{2}$	1	-1	-1	1
χ_n	1	0	$\frac{\varepsilon \mp \sqrt{\varepsilon s}}{2}$	$\frac{\varepsilon \pm \sqrt{\varepsilon s}}{2}$	1	-1	-1	1
$\chi(R^a)$	1	1	$\frac{(1+\varepsilon)(-1)^a}{2}$	$\frac{(1-\varepsilon)(-1)^a}{2}$	$r^a + r^{-a}$	$r_1^a + r_1^{-a}$	0	0
$\chi(S^b)$	1	-1	$-\frac{(1-\varepsilon)(-1)^b}{2}$	$-\frac{(1+\varepsilon)(-1)^b}{2}$	0	0	$-t^b - t^{-b}$	$-t_1^b - t_1^{-b}$

where r, r_1, t, t_1 , are the roots (except ± 1) of the respective equations $r^{\frac{s-1}{2}} = 1$, $r_1^{\frac{s-1}{2}} = -1$, $t^{\frac{s+1}{2}} = 1$, $t_1^{\frac{s+1}{2}} = -1$.

§ 6.

By the use of the following theorems, due to Frobenius, we are able to deduce the group-characters of F from those of H .

*If G is an invariant subgroup of the group H then every character of $\frac{H}{G}$ is also a character of H .**

*Über die Darstellung der endlichen Gruppen durch lineare Substitutionen, Berliner Sitzungsberichte, 1897, p. 995.

In order that a character of H may belong to the group $\frac{H}{G}$ it is necessary and sufficient that it have the same value for all elements of G . Then it has also equal values for every two elements of H which are equivalent mod. G .*

In the present case the invariant subgroup G of H is composed of the substitutions $(l) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and (λ) the identity. Then those and only those characters of H for which $\chi_l = \chi_\lambda$ belong to the group $\frac{H}{G} = F$; and since every character of F belongs to H , we obtain in this way all the characters of F . The classes (μ) and (m) , also (ν) and (n) , are equivalent mod. G . Hence we can write down at once the table of characters for F .

N	1	1	2	$\frac{s - \varepsilon - 4}{4}$	$\frac{s + \varepsilon - 2}{4}$
χ_λ	1	s	$\frac{s + \varepsilon}{2}$	$s + 1$	$s - 1$
χ_μ	1	0	$\frac{\varepsilon \pm \sqrt{\varepsilon s}}{2}$	1	-1
χ_ν	1	0	$\frac{\varepsilon \mp \sqrt{\varepsilon s}}{2}$	1	-1
$\chi(R^a)$	1	1	$\frac{(1 + \varepsilon)(-1)^a}{2}$	$r^a + r^{-a}$	0
$\chi(S^b)$	1	-1	$\frac{(1 - \varepsilon)(-1)^b}{2}$	0	$-t^b - t^{-b}$

where r and t are the roots (except ± 1) of the respective equations $r^{\frac{s-1}{2}} = 1$, $t^{\frac{s+1}{2}} = 1$.

II.

The Binary Linear Homogeneous Group H_1 in the $GF[2^n]$.

The order of H_1 is $h = 2^n(2^{2^n} - 1)$, and the determinant of each substitution is unity. The group is holoedrally isomorphic with the group of all binary linear fractional substitutions in the $GF[2^n]$.

*Über Relationen zwischen den Charakteren einer Gruppe und denen ihrer Untergruppen, Berliner Sitzungsberichte, 1898, p. 510.

We define $\kappa = \alpha + \delta$ as the invariant of the substitution $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

The substitution $R = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix}$, where ρ is a primitive root of the $GF[2^n]$, generates a cyclic group of order $s - 1$. R^a is conjugate to R^{-a} and is always distinct from it. Hence the powers of R represent $\frac{s-2}{2}$ classes, each containing $s(s+1)$ substitutions.

Let σ be a primitive root of $\sigma^{s+1} = 1$. The substitution $S = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix}$ is of period $s + 1$. S^b is conjugate to S^{-b} and is distinct from it; thus the powers of S represent $\frac{s}{2}$ classes, each containing $s(s-1)$ substitutions.

The substitution $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of period two and invariant zero is one of $s^2 - 1$ conjugate substitutions. We denote this class by (0) and the identity by (λ).

The total number of classes of conjugate substitutions is $s + 1$.

Below is given the table of group-characters.

N	1	1	$2^{n-1} - 1$	2^{n-1}
χ_λ	1	2^n	$2^n + 1$	$2^n - 1$
χ_0	1	0	1	-1
$\chi(R^a)$	1	1	$r^a + r^{-a}$	0
$\chi(S^b)$	1	-1	0	$-t^b - t^{-b}$

where r and t are the roots (except unity) of the respective equations $r^{s-1} = 1$, $t^{s+1} = 1$. As before $e = f$.

III.

The Binary Linear Fractional Group F_1 in the $GF[p^n]$, $p > 2$, of all Determinants not Zero.

The order of F_1 is $h = s(s^2 - 1)$. The substitutions will be supposed written in the normal form, i. e., of determinant unity or a particular not-square in the $GF[p^n]$.

We shall denote the determinant $\alpha\delta - \beta\gamma$ of the substitution $V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ by τ , where $\tau = 1$, or ν a particular not-square; and we shall call $\pm \frac{1}{2}(\alpha + \delta)$ the invariant of V .

If two substitutions have not the same determinant they are not conjugate. If two substitutions (neither the identity) have the same determinant and the same invariant, they are conjugate under E_1 .

By canonical form theory we find that all the substitutions of the group can be reduced to one or another of the following canonical forms:

- A) $R = \begin{pmatrix} \rho, & 0 \\ 0, & \rho^{-1}\tau \end{pmatrix}$, ρ a mark $\neq 0$ of the $GF[p^n]$;
 B) $S = \begin{pmatrix} \sigma, & 0 \\ 0, & \sigma^{-1}\tau \end{pmatrix}$, σ a mark $\neq 0$ of the $GF[p^{2n}]$;
 C) $T = \begin{pmatrix} 1, & 1 \\ 0, & 1 \end{pmatrix}$,

where σ satisfies a quadratic equation belonging to and irreducible in the $GF[p^n]$.

A) The substitution

$$R = \begin{pmatrix} \rho, & 0 \\ 0, & \rho^{-1}\nu \end{pmatrix}$$

where $\rho^2\nu^{-1}$ is a primitive root of the $GF[p^n]$, is of period $s-1$. With the exception of $R^{\frac{s-1}{2}}$ which is conjugate only to itself, R^a is conjugate to R^{-a} and is distinct from it. We have therefore $\frac{s-1}{2}$ classes represented by the powers of R , each containing $s(s+1)$ substitutions, except $R^{\frac{s-1}{2}}$, the class represented by which contains $\frac{1}{2}s(s+1)$ substitutions.

B) The group of all binary linear fractional substitutions in the $GF[p^n]$ of determinant $\neq 0$ is holodrically isomorphic with the group* of binary hyperorthogonal substitutions in the $GF[p^{2n}]$ of determinant a mark of the $GF[p^n]$ when taken fractionally, viz.,

$$U = \begin{pmatrix} A, & B \\ -\bar{B}, & \bar{A} \end{pmatrix} \quad (A\bar{A} + B\bar{B} = \pi),$$

where $\bar{A} \equiv A^*$ is the conjugate of A with respect to the $GF[p^n]$, and π is a mark $\neq 0$ of the $GF[p^n]$.

Consider the substitution

$$S = \begin{pmatrix} \sigma, & 0 \\ 0, & \sigma^{-1}\nu \end{pmatrix},$$

where σ is a primitive root of the equation $\sigma^{s+1} = \nu$. Since ν is an *arbitrary* not-square we may suppose that it is a primitive root of the $GF[p^n]$. Then σ is a

* Dickson, *Linear Groups*, § 144, Cor.

primitive root of the $GF[p^{2n}]$, and consequently S is of period $s + 1$. With the exception of $S^{\frac{s+1}{2}}$ which is conjugate only to itself, S^b is conjugate to S^{-b} and is distinct from it. We have therefore $\frac{s+1}{2}$ classes represented by the powers of S , each containing $s(s-1)$ substitutions, except $S^{\frac{s+1}{2}}$, the class represented by which contains $\frac{1}{2}s(s-1)$ substitutions.

The classes represented by the powers of $R(S)$ are characterized by the property that $\kappa^2 - \tau$ is a square (not-square) in the $GF[p^n]$, where $\tau = \nu$ or 1 according as the index is odd or even.

The substitution

$$T_\mu = \begin{pmatrix} 1, & \mu \\ 0, & 1 \end{pmatrix}, \quad \mu \text{ a mark } \neq 0 \text{ of the } GF[p^n],$$

of invariant ± 1 and determinant unity, is one of $s^2 - 1$ conjugate substitutions forming a class (μ) .

The total number of classes of conjugate substitutions is $s + 2$.

Below is given the table of group-characters.

N	1	1	1	1	$\frac{s-3}{2}$	$\frac{s-1}{2}$
χ_λ	1	1	s	s	$s+1$	$s-1$
χ_μ	1	1	0	0	1	-1
$\chi(R^{2a})$	1	1	1	1	$r^{2a} + r^{-2a}$	0
$\chi(S^{2b})$	1	1	-1	-1	0	$-t^{2b} - t^{-2b}$
$\chi(R^{2a+1})$	1	-1	1	-1	$r^{2a+1} + r^{-(2a+1)}$	0
$\chi(S^{2b+1})$	1	-1	-1	1	0	$-t^{2b+1} - t^{-(2b+1)}$

where r and t are the roots (except ± 1) of $r^{s-1} = 1$ and $t^{s+1} = 1$ respectively. As before $e = f$.

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VITA.

I was born near Lemonville, Ont., Can., and received my elementary education at Lemonville Public School and at Markham High School. In 1900 I received the degree of A. B., and in 1901 the degree of A. M., from McMaster University, Toronto, Ont. The following three years, with the exception of one quarter, I spent in the University of Chicago, in the departments of Mathematics and Astronomy, holding a fellowship in Mathematics 1901-1904.

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